Recap

- So far
  - Two ML algorithms: Naive Bayes, decision tree
  - Feature-based ML: Features interpretable and based on »domain knowledge«
  - Preprocessing
    - Convert into correct format
    - Transform features (e.g., remove rare feature values, normalize scales)
- Last Tuesdays
  - Training and test for a concrete example
  - Preprocessing pipeline needs to run on both

⚠ Next week: No Thursday!

Logistic Regression

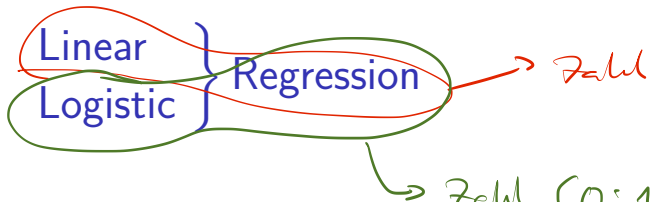
Sprachverarbeitung (VL + Ü)

Nils Reiter

June 15, 2023

INSTITUT FÜR
DIGITAL HUMANITIES
UNIVERSITÄT ZU KÖLN

Clustering → Gruppen
Klassifikation → Kategorien
Regression → Zahl



Regression

Linear regression

- Prediction of numeric values (e.g., COVID-19 cases)

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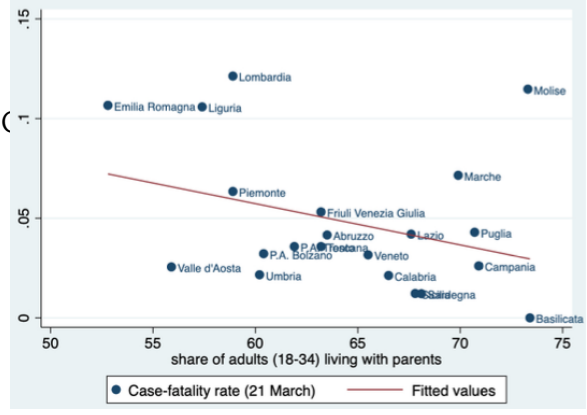

Regression

Linear regression

- Prediction of numeric values (e.g., COVID-19 cases)
- »Linear« regression: Prediction of a linear relation (i.e., a line)
- Most real problems are not linear – in particular not COVID-19 cases ...

Regression

Linear regression

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- Most real problems are not linear – in particular not COVID-19 cases ...

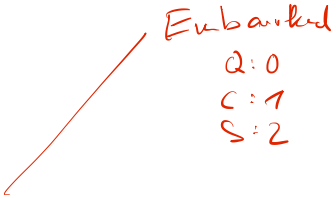
Logistic Regression

- Classification algorithm: Instances are grouped into *previously known* classes
- Binary classification: Two classes (e.g., positive/negative)
- Extension of linear regression

Linear/logistic regression in parallel

Linear Regression

Task Setup



- Input (x): A (collection of) numeric feature values
- Output (y): A numeric value

Example

Given the length of a narrative text in words, predict the number of characters present in its plot

Linear Regression

The data set

x	y (# characters)
101	3
105	5
150	8
210	12
250	7
295	13

Linear Regression

The data set

x	y (# characters)
10	3
105	5
150	8
210	12
250	7
295	13

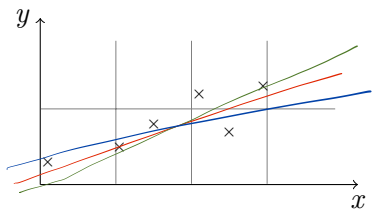


Figure: Data set, each $\times$ represents a text (x: text length, y: num. of characters)

Linear Regression

The data set

x	y (# characters)
10	3
105	5
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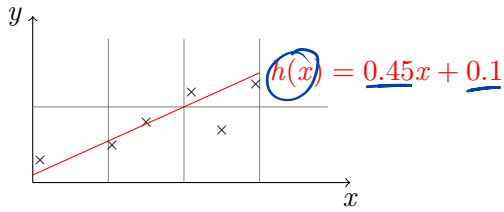
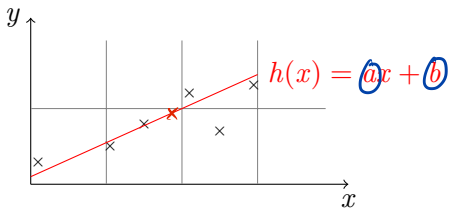


Figure: Data set, each $\times$ represents a text (x: text length, y: num. of characters)

Linear Regression



Prediction Model

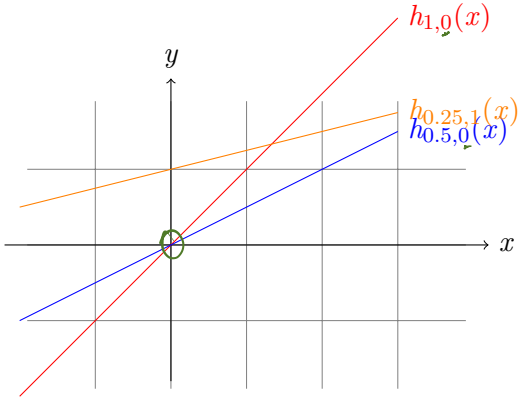
- Linear regression with one variable (= univariate linear regression)
- Data: (x, y)
- Prediction (hypothesis function): $y = h_{a,b}(x) = ax + b$
- How to set parameters a and b? $\rightarrow$ training algorithm

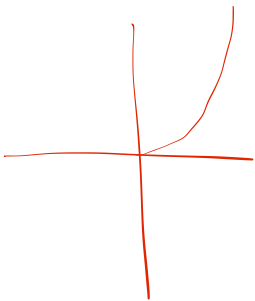
Linear Regression

Prediction Model

b: y-Achsenabschnitt

- $h_{a,b}(x) = ax + b$ describes a set of functions
  - $h_{1,0}(x)$ is one concrete function





Linear vs. Logistic Regression

- Linear regression: Prediction of numerical data
- Logistic regression: Prediction of (binary) categorical data

Linear vs. Logistic Regression

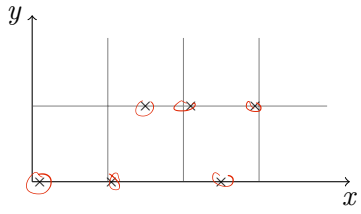
- Linear regression: Prediction of numerical data
- Logistic regression: Prediction of (binary) categorical data

Examples

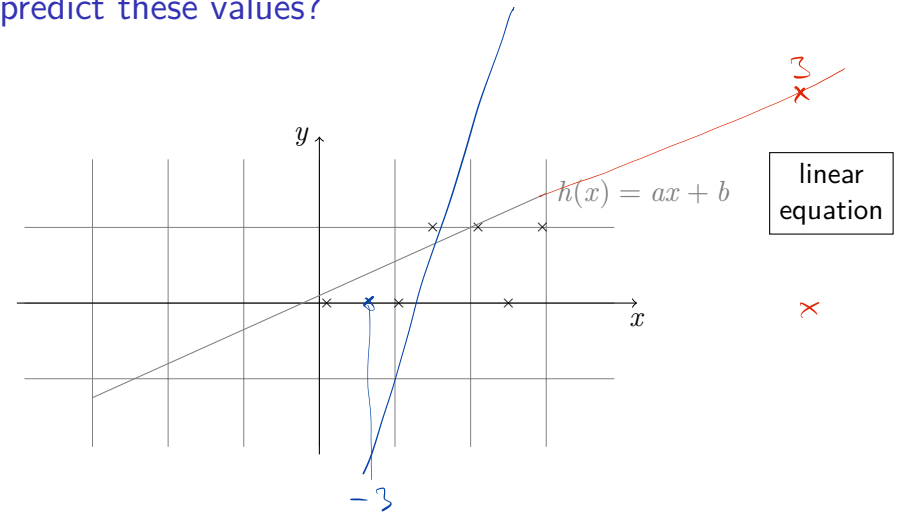
- Our interest
  - Literature quality
- Given the number of characters in a narrative text
- Will a book win the Nobel prize?
  - Two classes: Yes/No

Logistic Regression

The data set



How to predict these values?



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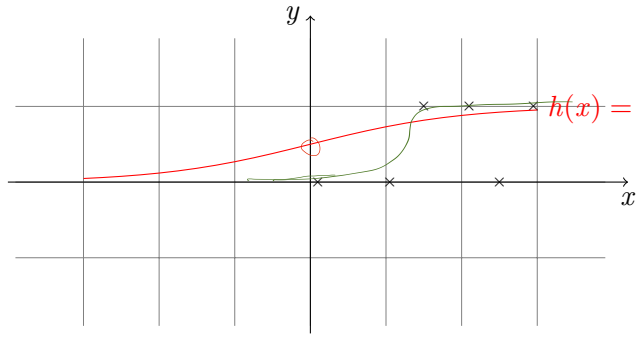
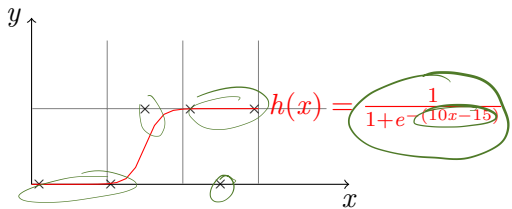


$h(x) = \frac{1}{1+e^{-x}}$

logistic function

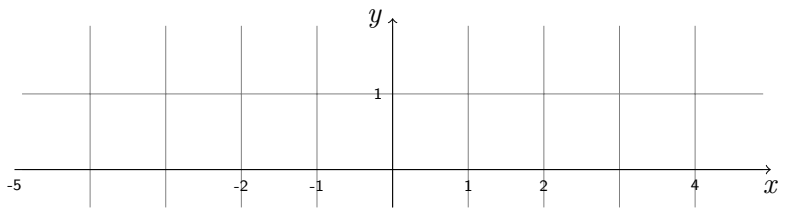
Parameter Fitting

$$\frac{1}{1+e^{-(ax+b)}}$$



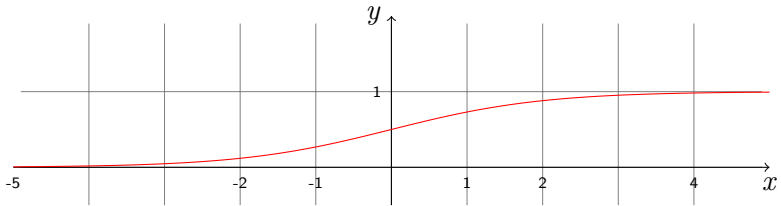
- Linear equations can be wrapped in a logistic one
- Same parameters to be tuned (a and b)
- $e = \sum_{n=0}^{\infty} \frac{1}{n!} = 2.71828$ (Euler's number)

The Logistic Function



$y = \frac{1}{1+e^{-(ax+b)}}$ (general form)

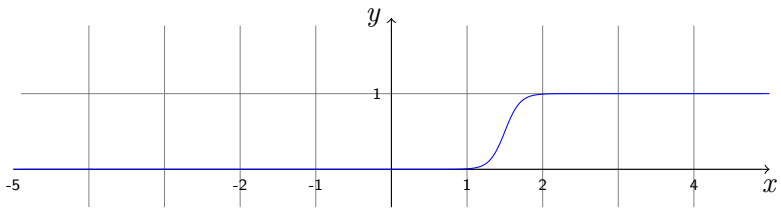
The Logistic Function



$$y = \frac{1}{1+e^{-(ax+b)}} \qquad \text{(general form)}$$

$$y = \frac{1}{1+e^{-(1*x+0)}} = \frac{1}{1+e^{-x}}$$

The Logistic Function



$$y = \frac{1}{1+e^{-(ax+b)}} \qquad \text{(general form)}$$

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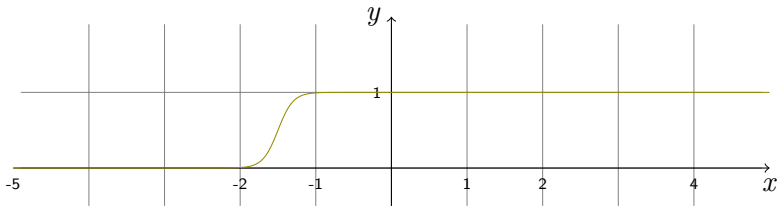
The Logistic Function



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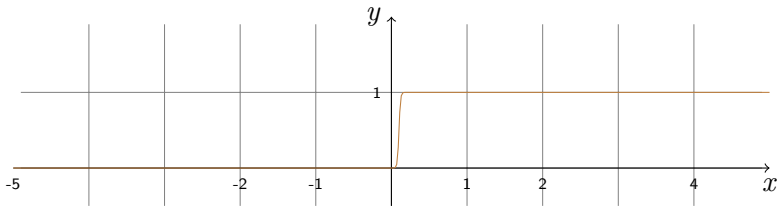
The Logistic Function



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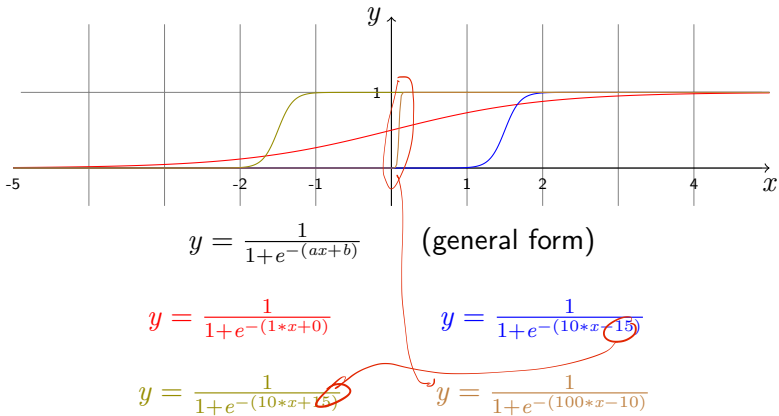
$$y = \frac{1}{1+e^{-(10*x+15)}} \qquad y = \frac{1}{1+e^{-(100*x-10)}}$$

The Logistic Function



Summary: Logistic Regression (with a single variable)

Logistic regression is half of the math of deep learning

Summary: Logistic Regression (with a single variable)

Logistic regression is half of the math of deep learning

- Logistic Regression: Predicting binary values
- Model ⟶ Vorhersagemodell
  - Logistic equations
  - $y = \frac{1}{1+e^{-(ax+b)}}$
- Learning algorithm: How to choose a and b?

Gradient Descent

Learning Regression Models

- How to select the parameters a, b such that the hypothesis function describes the data points as best as possible?
- Learning algorithm *Gradient Descent*

Learning Regression Models

- How to select the parameters a, b such that the hypothesis function describes the data points as best as possible?
- Learning algorithm *Gradient Descent*

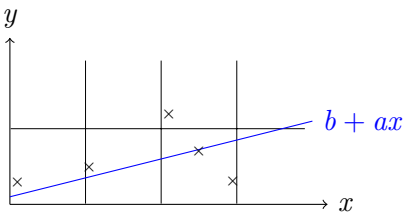
$a = 7$

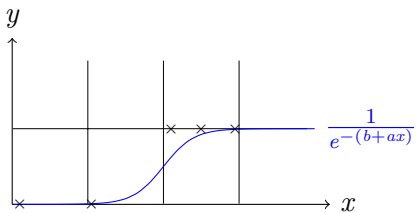
$b = 3$

Gradient descent is half of the algorithms of deep learning

Loss: Intuition

The *loss* measures the ›wrongness‹ of values for a and b.





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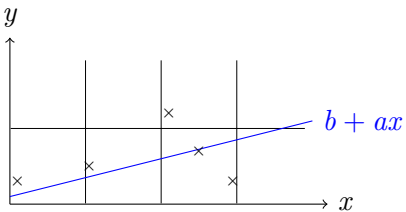

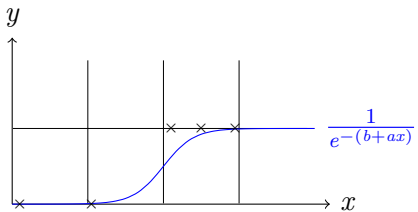

- How big is the gap between a hypothesis and the data?
- Is $(a, b) = (0.3, 0.5)$ or $(a, b) = (0.4, 0.4)$ better?

Loss: Intuition

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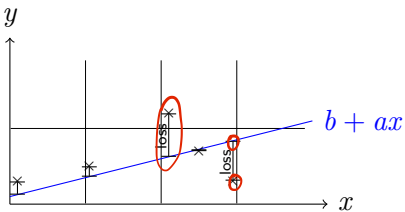

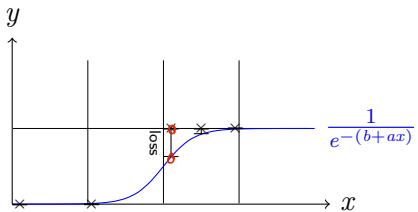

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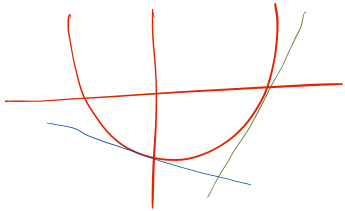
Loss function: Intuition

[5, 3]

- Loss should be as small as possible
- Total loss can be calculated for given parameters $\vec{w} = (a, b)$ (and a full data set)
  $\Rightarrow$ I.e.: Loss can be expressed as a function of $\vec{w}$!

Loss function: Intuition



- Loss should be as small as possible
- Total loss can be calculated for given parameters $\vec{w} = (a, b)$ (and a full data set)
  - $\Rightarrow$ I.e.: Loss can be expressed as a function of $\vec{w}$!
- Idea:
  - We change $\vec{w}$ until we find the minimum of the function
  - We use the derivative to find out if we are in a minimum
  - The derivative also tells us how to change the update parameters a and b

Loss Function: Intuition

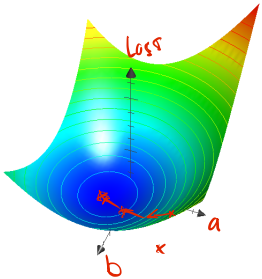
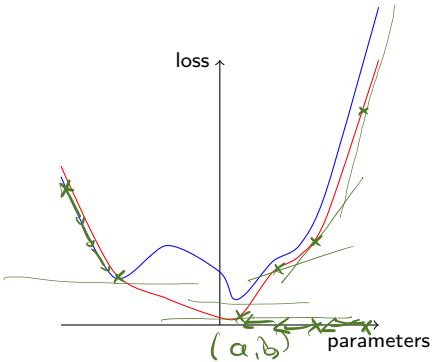
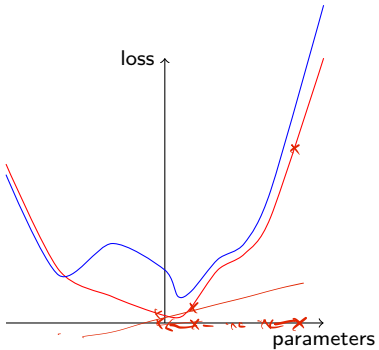


Figure: The loss function with two parameters

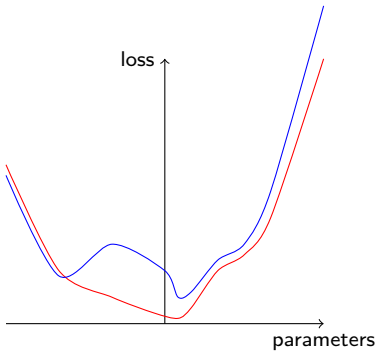
Loss function: Intuition



Loss function: Intuition



Loss function: Intuition



Function should be convex!
If not, we might get stuck in local minimum

Hypothesis vs. Loss Function

$h(x) = ax + b$

$J_D(\vec{w}) = \dots$

- Hypothesis function h
  - Calculates outcomes, given feature values x
- Loss function J
  - Calculates ›wrongness‹ of h, given parameter values $\vec{w}$ (and a data set)
  - In reality, $\vec{w}$ represents many more parameters (thousands)

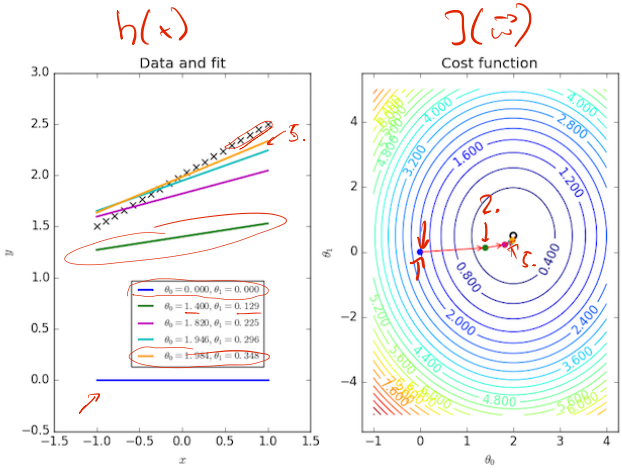


Figure: Visualizing gradient descent Source

Loss Function

Definition

Loss function depends on hypothesis function

Linear hypothesis function

- $h(x) = ax + b$
- Loss: Mean squared error

Loss Function

Definition

Loss function depends on hypothesis function

Linear hypothesis function

- $h(x) = ax + b$
- Loss: Mean squared error

Logistic hypothesis function

- $h(x) = \frac{1}{e^{-(b+ax)}}$
- Loss: (Binary) cross-entropy loss

Loss Function

Definition for Linear Regression

- The loss function is a function on parameter values a and b
  (for a given hypothesis function and data set)
  - Hypothesis function: $h_{\vec{w}} = w_1 x + w_0$

$\vec{w} = (a, b)$: parameters $h_{\vec{w}}$: hypothesis function m: number of items

$$J(\vec{w}) =$$

Loss Function

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$$J(\vec{w}) = h_{\vec{w}}(x_i) - y_i$$

korrekt

hypothese

- Calculate the loss for item i

Loss Function

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$$J(\vec{w}) = \qquad (h_{\vec{w}}(x_i) - y_i)^2$$

- Calculate the loss for item i
- Square the error

Loss Function

Definition for Linear Regression

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$\vec{w} = (a, b)$: parameters $\quad h_{\vec{w}}$: hypothesis function $\quad m$: number of items

$$J(\vec{w}) = \sum_{i=1}^{m} (h_{\vec{w}}(x_i) - y_i)^2$$

- Calculate the loss for item i
- Square the error
- Sum them up

Loss Function

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$\vec{w} = (a, b)$: parameters $h_{\vec{w}}$: hypothesis function m: number of items

$$J(\vec{w}) = \frac{1}{m} \sum_{i=1}^{m} (h_{\vec{w}}(x_i) - y_i)^2$$

- Calculate the loss for item i
- Square the error
- Sum them up
- Divide by the number of items
  - Known as: *Mean squared error*

Loss Function

Definition for Linear Regression

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  (for a given hypothesis function and data set)
  - Hypothesis function: $h_{\vec{w}} = w_1 x + w_0$

$\vec{w} = (a, b)$: parameters $h_{\vec{w}}$: hypothesis function m: number of items

$$J(\vec{w}) = \frac{1}{2} \frac{1}{m} \sum_{i=1}^{m} (h_{\vec{w}}(x_i) - y_i)^2$$

- Calculate the loss for item i
- Square the error
- Sum them up
- Divide by the number of items
  - Known as: *Mean squared error*
- Divide by two
  - out of convenience, because derivation

Loss function

Definition for Logistic Regression

- Two cases: $y_i = 0$ or $y_i = 1$ – y_i: real outcome for instance i

Loss function

Definition for Logistic Regression

- Two cases: $y_i = 0$ or $y_i = 1$ – y_i: real outcome for instance i

$$J(\vec{w}) = \qquad h_{\vec{w}}(x_i) + \qquad (1 - h_{\vec{w}}(x_i))$$

Loss function

Definition for Logistic Regression

- Two cases: $y_i = 0$ or $y_i = 1 - y_i$: real outcome for instance i

$$J(\vec{w}) = \qquad \log h_{\vec{w}}(x_i) + \qquad \log\left(1 - h_{\vec{w}}(x_i)\right)$$

y_i	$h_{\vec{w}}(x_i)$	$y_i \log h_{\vec{w}}(x_i) + (1 - y_i) \log(1 - h_{\vec{w}}(x_i))$
0	1	-23.2535
0	0	0

Loss function

Definition for Logistic Regression

- Two cases: $y_i = 0$ or $y_i = 1 - y_i$: real outcome for instance i

$$J(\vec{w}) = \qquad y_i \log h_{\vec{w}}(x_i) + (1 - y_i) \log \left(1 - h_{\vec{w}}(x_i)\right)$$

y_i	$h_{\vec{w}}(x_i)$	$y_i \log h_{\vec{w}}(x_i) + (1 - y_i) \log(1 - h_{\vec{w}}(x_i))$
0	1	-23.2535
0	0	0
1	1	0
1	0	-23.2535

Loss function

Definition for Logistic Regression

- Two cases: $y_i = 0$ or $y_i = 1 - y_i$: real outcome for instance i

$$J(\vec{w}) = -\frac{1}{m}\sum_{i=0}^{m} y_i \log h_{\vec{w}}(x_i) + (1 - y_i) \log\left(1 - h_{\vec{w}}(x_i)\right)$$

y_i	$h_{\vec{w}}(x_i)$	$y_i \log h_{\vec{w}}(x_i) + (1 - y_i) \log(1 - h_{\vec{w}}(x_i))$
0	1	-23.2535
0	0	0
1	1	0
1	0	-23.2535
1	0.8	-0.3219281
1	0.2	-2.321928

Loss function

Definition for Logistic Regression

- Two cases: $y_i = 0$ or $y_i = 1 - y_i$: real outcome for instance i

$$J(\vec{w}) = -\frac{1}{m}\sum_{i=0}^{m} \underbrace{y_i \log h_{\vec{w}}(x_i)}_{0 \text{ iff } y_i=0} + \underbrace{(1-y_i)\log(1-h_{\vec{w}}(x_i))}_{0 \text{ iff } y_i=1}$$

y_i	$h_{\vec{w}}(x_i)$	$y_i \log h_{\vec{w}}(x_i) + (1-y_i)\log(1-h_{\vec{w}}(x_i))$
0	1	-23.2535
0	0	0
1	1	0
1	0	-23.2535
1	0.8	-0.3219281
1	0.2	-2.321928

Caveat: $\log 0$ is undefined
We may need to add something very small

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  - No information loss

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  - No information loss
- Multiplication turns to addition: $\log(a \cdot b) = \log a + \log b$
  - Addition is much faster than multiplication in a computer
  - Pays off because we're doing this *a lot*

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- Above: 1 dimension, 2 parameters
  - a: slope, b: y-intercept
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  - Input vector $\vec{x}$ with $n - 1$ dimensions
  - Hypothesis function: $h_{\vec{w}}(x) = w_n x_n + w_{n-1} x_{n-1} + \ldots w_1 x_1 + w_0$
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- Algorithms
  - Derivatives more complicated
  - Otherwise identical

Section 2

Summary

Summary

Regression

- Fitting parameters to a data distribution
  - Linear R: Numeric prediction algorithm
    - Prediction model: $h_{\vec{w}}(x) = ax + b$
  - Logistic R: Classification algorithm
    - Prediction model: $h_{\vec{w}}(x) = \frac{1}{e^{-(b+ax)}}$
- Learning algorithm: Gradient descent

Gradient Descent

- Initialise $\vec{w}$ with random values (e.g., 0)
- Repeat:
  - Find the direction to the minimum by taking the derivative
  - Change $\vec{w}$ accordingly, using a learning rate η
  - Stop when $\vec{w}$ don't change anymore