UNIVERSITÄT
ZU KÖLN

Machine Learning 1: Naive Bayes

VL Sprachliche Informationsverarbeitung

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INSTITUT FÜR
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Introduction

- Probabilistic classification algorithm
- Makes independence assumption about features – ‘naive’
- Reading Jurafsky/Martin (2023, Ch. 4)

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- Reading — Jurafsky/Martin (2023, Ch. 4)
- Nice intro to Bayesian statistics by Matt Parker and Hannah Fry — Parker/Fry (2019)

Section 1

Probabilities

Basics: Cards

- 32 cards Ω (sample space)
- 4 'colors': $C = \{\clubsuit, \spadesuit, \diamondsuit, \heartsuit\}$
- 8 values: $V = \{7, 8, 9, 10, J, Q, K, A\}$
- Individual cards ('outcomes') are denoted with value and color: $8\heartsuit$

Basics

Events

- Generally, we draw cards from a (well shuffled) deck
- We define what events we are interested in
- An event can be any subset of the sample space Ω
  - There are $2^{|\Omega|}$ different subsets, i.e., $2^{|\Omega|}$ possible events
- Events will be denoted with E

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- "We draw a heart eight" – $E = \{8\heartsuit\}$
- "We draw card with a diamond"

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- "We draw a heart eight" – $E = \{8\heartsuit\}$
- "We draw card with a diamond" – $E = \{7\diamondsuit, 8\diamondsuit, 9\diamondsuit, 10\diamondsuit, J\diamondsuit, Q\diamondsuit, K\diamondsuit, A\diamondsuit\}$

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- "We draw card with a diamond" – $E = \{7\diamondsuit, 8\diamondsuit, 9\diamondsuit, 10\diamondsuit, J\diamondsuit, Q\diamondsuit, K\diamondsuit, A\diamondsuit\}$
- "We draw a queen"

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- "We draw card with a diamond" – $E = \{7\diamondsuit, 8\diamondsuit, 9\diamondsuit, 10\diamondsuit, J\diamondsuit, Q\diamondsuit, K\diamondsuit, A\diamondsuit\}$
- "We draw a queen" – $E = \{Q\clubsuit, Q\spadesuit, Q\diamondsuit, Q\heartsuit\}$

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- "We draw a queen" – $E = \{Q\clubsuit, Q\spadesuit, Q\diamondsuit, Q\heartsuit\}$
- "We draw a heart eight or diamond ten"

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- "We draw a queen" – $E = \{Q\clubsuit, Q\spadesuit, Q\diamondsuit, Q\heartsuit\}$
- "We draw a heart eight or diamond ten" – $E = \{8\heartsuit, 10\diamondsuit\}$
- "We draw any card"

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- "We draw a heart eight" – $E = \{8\heartsuit\}$
- "We draw card with a diamond" – $E = \{7\diamondsuit, 8\diamondsuit, 9\diamondsuit, 10\diamondsuit, J\diamondsuit, Q\diamondsuit, K\diamondsuit, A\diamondsuit\}$
- "We draw a queen" – $E = \{Q\clubsuit, Q\spadesuit, Q\diamondsuit, Q\heartsuit\}$
- "We draw a heart eight or diamond ten" – $E = \{8\heartsuit, 10\diamondsuit\}$
- "We draw any card" – $E = \Omega$

Basics

Probabilities

- Probability $p(E)$: Likelihood, that a certain event ($E \subset \Omega$) happens
  - $0 \leq p \leq 1$
  - $p(E) = 0$: Impossible event $p(E) = 1$: Certain event
  - $p(E) = 0.000001$: Very unlikely event

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Example

- If all outcomes are equally likely: $p(E) = \frac{|E|}{|\Omega|}$
- $p(\{8\heartsuit\}) = \frac{1}{32}$
- $p(\{9\clubsuit, 9\spadesuit, 9\diamondsuit, 9\heartsuit\}) = \frac{4}{32}$
- $p(\Omega) = 1$ (must happen, certain event)

Basics

Probability and Relative Frequency

- Probability p: Theoretical concept, idealisation
  - Expectation
- Relative Frequency f: Concrete measure
  - Normalised number of *observed* events
  - E.g., after 10 times drawing a card (with returning and shuffling), we counted the event ♠ eight times: $f(\{x\spadesuit\}) = \frac{8}{10}$
- For large numbers of drawings, relative frequency approximates the probability
  - $\lim_{\infty} f = p$

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- For large numbers of drawings, relative frequency approximates the probability
  - $\lim_{\infty} f = p$
- In practice, we will often use relative frequencies as probabilities
- This establishes assumptions:
  - Data set is representative of the real world
  - We make a lot of observations (the more, the better we approximate real probabilities)

Basics

Joint Probability (Independent Events)

- We are often interested in multiple events (and their relation)
- E: We draw $8\heartsuit$ two times in a row (putting the first card back)
  - E_1: First card is $8\heartsuit$
  - E_2: Second card is $8\heartsuit$
  - $p(E) = p(E_1, E_2) = p(E_1) * p(E_2) = \frac{1}{32} * \frac{1}{32} = 0.0156$

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- E: We draw $\heartsuit$ two times in a row (putting the first card back)
  - E_1: First card is $X\heartsuit$
  - E_2: Second card is $X\heartsuit$
  - $p(E) = p(E_1, E_2) = p(E_1) * p(E_2) = \frac{1}{4} * \frac{1}{4} = 0.0625$

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  - $p(E) = p(E_1, E_2) = p(E_1) * p(E_2) = \frac{1}{4} * \frac{1}{4} = 0.0625$
- These events are independent
  - because we return and re-shuffle the cards all the time
  - Drawing $8\heartsuit$ the first time has no influence on the second drawing

Basics I

Conditional Probability (Dependent Events)

- We no longer return the card
- E: We draw $8\heartsuit$ two times in a row
  - E_1: First card is $8\heartsuit$
  - E_2: Second card is $8\heartsuit$ (without putting the first card back)
  - $\cancel{p(E_1, E_2) = p(E_1) * p(E_2)}$
  - This no longer works, because the events are not independent
  - There is only one $8\heartsuit$ in the game, and $p(E_2)$ has to take into account that it might be gone already
  - This is expressed with the notion of conditional probability
  - $p(E_1, E_2) = p(E_1) * p(E_2|E_1)$
    - $p(E_2|E_1) = 0$, therefore $p(E_1, E_2) = 0$

Basics II

Conditional Probability (Dependent Events)

- E: We draw $\heartsuit$ first (E_1), followed by:
  - E_2: Second card is $X\diamondsuit$
  - E_3: Second card is $X\heartsuit$
  - $p(E_1, E_2) = p(E_1) * p(E_2|E_1) = \frac{8}{32} * \frac{8}{31} = 0.064$
  - $p(E_1, E_3) = p(E_1) * p(E_3|E_1) = \frac{8}{32} * \frac{7}{31} = 0.056$

Conditional and Joint Probabilities

Example

Relation between **hair color** H and preferred **wake-up time** W (all numbers are made up.)

$\downarrow W \ / \ H \rightarrow$	brown	red	sum
early	20	10	30
late	30	5	35
sum	50	15	65

Table: Experimental Results, Ω: Group of questioned people, $|\Omega| = 65$

Conditional and Joint Probabilities

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- If we pick a random person, what's the probability that this person has brown hair?

$$p(H = \text{brown}) = ?$$

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$$\left.\begin{array}{ll} p(H = \text{brown}) = \frac{50}{65} & p(H = \text{red}) = \frac{15}{65} \\ p(W = \text{early}) = \frac{30}{65} & p(W = \text{late}) = \frac{35}{65} \end{array}\right\} \text{sums per row or column}$$

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- Joint probability: $p(W = \text{late}, H = \text{brown}) = \frac{30}{65}$
  - Probability that someone has brown hair *and* prefers to wake up late
  - Denominator: Number of all items

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- Joint probability: $p(W = \text{late}, H = \text{brown}) = \frac{30}{65}$
  - Probability that someone has brown hair *and* prefers to wake up late
  - Denominator: Number of all items
- Conditional probability: $p(W = \text{late} | H = \text{brown}) = \frac{30}{50}$
  - Probability that one of the brown-haired participants prefers to wake up late
  - Denominator: Number of remaining items (after conditioned event has happened)

Conditional and Joint Probabilities

Example

	brown	red	margin
early	$p(W = e, H = b) = 0.31$	$p(W = e, H = r) = 0.15$	$p(W = e) = 0.46$
late	$p(W = l, H = b) = 0.46$	$p(W = l, H = r) = 0.08$	$p(W = l) = 0.54$
margin	$p(H = b) = 0.77$	$p(H = r) = 0.23$	$p(\Omega) = 1$

Table: (Joint) Probabilities, derived by dividing everything by $|\Omega|$

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$$p(A|B) \quad = \quad \frac{p(A, B)}{p(B)} \quad \text{definition of conditional probabilities}$$

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p(A|B) &= \frac{p(A,B)}{p(B)} && \text{definition of conditional probabilities} \\
p(W = \text{late}|H = \text{brown}) &= \frac{30}{50} = 0.6 && \text{intuition from previous slide}
\end{aligned}
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&= \frac{p(W = \text{late}, H = \text{brown})}{p(H = \text{brown})} && \text{by applying definition}
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&= \frac{p(W=\text{late}, H=\text{brown})}{p(H=\text{brown})} && \text{by applying definition} \\
&= \frac{0.46}{0.77} = 0.6
\end{aligned}
$$

Multiple Conditions

- Joint probabilities can include more than two events
  $p(E_1, E_2, E_3, \dots)$
- Conditional probabilities can be conditioned on more than two events

$$p(A|B, C, D) = \frac{p(A, B, C, D)}{p(B, C, D)}$$

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$$p(A|B, C, D) = \frac{p(A, B, C, D)}{p(B, C, D)}$$

- Chain rule

$$\begin{aligned} p(A, B, C, D) &= p(A|B, C, D)p(B, C, D) \\ &= p(A|B, C, D)p(B|C, D)p(C, D) \\ &= p(A|B, C, D)p(B|C, D)p(C|D)p(D) \end{aligned}$$

Bayes Law

$$p(B|A) = \frac{p(A,B)}{p(A)} = \frac{p(A|B)p(B)}{p(A)}$$

Allows reordering of conditional probabilities

- Follows directly from above definitions

Section 2

Naive Bayes

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- Probabilities are estimated on training data (relative frequencies)

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  - A data set $x \in X$ (x is an individual instance, X the entire set)
  - The feature *value* is given as $f_i(x)$

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Example

- Feature representing "word length" f_6
- One data point is "dog"
- $f_6(\text{``}dog\text{''}) = 6$

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Example

- Feature representing "word length" f_6
- One data point is "dog"
- $f_6(\text{“}dog\text{”}) = 6$

You can also think of f_6 as a function in a program:

```
def f6(x):
  return len(x)
```

Naive Bayes

Prediction Model

Intuition

We calculate the probability for each possible class c, given the feature values of the item x, and we assign most probably class

Naive Bayes

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Intuition

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- $f_n(x)$: Value of feature n for instance x
- $\arg\max_i e$: Select the argument i that maximizes the expression e

Naive Bayes

Prediction Model

```
def argmax(SET, EXPRESSION):
  arg = 0
  maxvalue = 0
  foreach i in SET:
    value = EXPRESSION(i)
    if value > maxvalue:
      arg = i
      maxvalue = value
  return arg
```

Intuition

We calculate the probability for each possible class c, given the and we assign most probably class

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$$\text{prediction}(x) = \arg\max_{c \in C} p(c|f_1(x), f_2(x), \ldots, f_n(x))$$

Naive Bayes

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$$\text{prediction}(x) = \arg\max_{c \in C} p(c|f_1(x), f_2(x), \ldots, f_n(x))$$

How do we calculate $p(c|f_1(x), f_2(x), \ldots, f_n(x))$?

Naive Bayes
Prediction Model

$$p(c|f_1, \ldots, f_n) \quad =$$

Naive Bayes

Prediction Model

$$p(c|f_1, \ldots, f_n) \quad = \quad \frac{p(c, f_1, f_2, \ldots, f_n)}{p(f_1, f_2, \ldots, f_n)}$$

Naive Bayes

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$$p(c|f_1,\ldots,f_n) \quad = \quad \frac{p(c,f_1,f_2,\ldots,f_n)}{p(f_1,f_2,\ldots,f_n)} = \frac{p(f_1,f_2,\ldots,f_n,c)}{p(f_1,f_2,\ldots,f_n)}$$

Naive Bayes

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\begin{aligned}
p(c|f_1,\ldots,f_n) &= \frac{p(c,f_1,f_2,\ldots,f_n)}{p(f_1,f_2,\ldots,f_n)} = \frac{p(f_1,f_2,\ldots,f_n,c)}{p(f_1,f_2,\ldots,f_n)} \\
& \quad \text{denominator is constant, so we skip it} \\
&\propto p(f_1|f_2,\ldots,f_n,c) \times p(f_2|f_3,\ldots,f_n,c) \times \cdots \times p(c)
\end{aligned}
$$

Naive Bayes

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$$\propto p(f_1|f_2,\ldots,f_n,c) \times p(f_2|f_3,\ldots,f_n,c) \times \cdots \times p(c)$$

Now we – naively – assume feature independence

$$= p(f_1|c) \times p(f_2|t) \times \cdots \times p(c)$$

Naive Bayes

Prediction Model

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\begin{aligned}
p(c|f_1,\ldots,f_n) &= \frac{p(c,f_1,f_2,\ldots,f_n)}{p(f_1,f_2,\ldots,f_n)} = \frac{p(f_1,f_2,\ldots,f_n,c)}{p(f_1,f_2,\ldots,f_n)} \\
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&\propto p(f_1|f_2,\ldots,f_n,c) \times p(f_2|f_3,\ldots,f_n,c) \times \cdots \times p(c) \\
& \quad \text{Now we – naively – assume feature independence} \\
&= p(f_1|c) \times p(f_2|t) \times \cdots \times p(c) \\
\mathrm{prediction}(x) &= \arg\max_{c\in C} p(f_1(x)|c) \times p(f_2(x)|c) \times \cdots \times p(c)
\end{aligned}
$$

Naive Bayes

Prediction Model

$$p(c|f_1,\ldots,f_n) = \frac{p(c,f_1,f_2,\ldots,f_n)}{p(f_1,f_2,\ldots,f_n)} = \frac{p(f_1,f_2,\ldots,f_n,c)}{p(f_1,f_2,\ldots,f_n)}$$

denominator is constant, so we skip it

$$\propto p(f_1|f_2,\ldots,f_n,c) \times p(f_2|f_3,\ldots,f_n,c) \times \cdots \times p(c)$$

Now we – naively – assume feature independence

$$= p(f_1|c) \times p(f_2|t) \times \cdots \times p(c)$$

$$\mathrm{prediction}(x) = \arg\max_{c\in C} p(f_1(x)|c) \times p(f_2(x)|c) \times \cdots \times p(c)$$

$\boxed{\text{Where do we get } p(f_i(x)|c)\text{? – Training!}}$

Naive Bayes

Learning Algorithm

1. For each feature f_i
   - Count frequency tables from the training set:

		C (classes)			
		c_1	c_2	...	c_m
$v(f_i)$	a	3	2	...	
	b	5	7	...	
	c	0	1	...	
	$\sum$	8	10		

2. Calculate conditional probabilities
   - Divide each number by the sum of the entire column
     - E.g., $p(a|c_1) = \frac{3}{3+5+0}$ $\quad p(b|c_2) = \frac{7}{2+7+1}$

Section 3

Example: Spam Classification

Training

- Data set: 100 e-mails, manually classified as spam or not spam (50/50)
  - Classes $C = \{\text{true}, \text{false}\}$
- Features: Presence of each of these tokens (manually selected): 'casino', 'enlargement', 'meeting', 'profit', 'super', 'text', 'xxx'

		C	
		true	false
casino	1	45	25
	0	5	25
	$\sum$	50	50

		C	
		true	false
text	1	15	35
	0	35	15
	$\sum$	50	50

...

Table: Extracted frequencies for features 'casino' and 'text'

Prediction

1. Extract word presence information from new text
2. Calculate the probability for *each possible class*

$$p\left(\text{true} \,\middle|\, \begin{bmatrix} \text{casino} & 0 \\ \text{enlargement} & 0 \\ \text{meeting} & 1 \\ \text{profit} & 0 \\ \text{super} & 0 \\ \text{text} & 1 \\ \text{xxx} & 1 \end{bmatrix}\right)$$

Prediction

1. Extract word presence information from new text
2. Calculate the probability for *each possible class*

$$p\left(\text{true} \middle| \begin{bmatrix} \text{casino} & 0 \\ \text{enlargement} & 0 \\ \text{meeting} & 1 \\ \text{profit} & 0 \\ \text{super} & 0 \\ \text{text} & 1 \\ \text{xxx} & 1 \end{bmatrix}\right) \propto \begin{array}{ll} p(\text{casino} = 0|\text{true}) & \times \\ p(\text{enlargement} = 0|\text{true}) & \times \\ p(\text{meeting} = 1|\text{true}) & \times \\ p(\text{profit} = 0|\text{true}) & \times \\ p(\text{super} = 0|\text{true}) & \times \\ p(\text{text} = 1|\text{true}) & \times \\ p(\text{xxx} = 1|\text{true}) & \end{array}$$

Prediction

1. Extract word presence information from new text
2. Calculate the probability for *each possible class*

$$p\left(\text{true}\;\middle|\;\begin{bmatrix} \text{casino} & 0 \\ \text{enlargement} & 0 \\ \text{meeting} & 1 \\ \text{profit} & 0 \\ \text{super} & 0 \\ \text{text} & 1 \\ \text{xxx} & 1 \end{bmatrix}\right) \propto \begin{array}{ll} p(\text{casino} = 0|\text{true}) & \times \\ p(\text{enlargement} = 0|\text{true}) & \times \\ p(\text{meeting} = 1|\text{true}) & \times \\ p(\text{profit} = 0|\text{true}) & \times \\ p(\text{super} = 0|\text{true}) & \times \\ p(\text{text} = 1|\text{true}) & \times \\ p(\text{xxx} = 1|\text{true}) & \end{array}$$

$$= \quad \cdots \times \frac{5}{50} \times \cdots \times \frac{15}{50} \times \cdots = \ldots$$

Prediction

1. Extract word presence information from new text
2. Calculate the probability for *each possible class*

$$p\left(\text{true} \middle| \begin{bmatrix} \text{casino} & 0 \\ \text{enlargement} & 0 \\ \text{meeting} & 1 \\ \text{profit} & 0 \\ \text{super} & 0 \\ \text{text} & 1 \\ \text{xxx} & 1 \end{bmatrix}\right) \propto \begin{array}{ll} p(\text{casino} = 0|\text{true}) & \times \\ p(\text{enlargement} = 0|\text{true}) & \times \\ p(\text{meeting} = 1|\text{true}) & \times \\ p(\text{profit} = 0|\text{true}) & \times \\ p(\text{super} = 0|\text{true}) & \times \\ p(\text{text} = 1|\text{true}) & \times \\ p(\text{xxx} = 1|\text{true}) & \end{array}$$

$$= \quad \cdots \times \frac{5}{50} \times \cdots \times \frac{15}{50} \times \cdots = \ldots$$

$$p\left(\text{false} \middle| \begin{bmatrix} \text{casino} & 0 \\ \vdots & \vdots \end{bmatrix}\right) \propto \quad \ldots$$

3. Assign the class with the higher probability

Subsection 1

Problems with Zeros

Danger

		C	
		true	false
love	1	0	35
	0	50	15
	$\sum$	50	50

- What happens in this situation to the prediction?

Danger

		C	
		true	false
love	1	0	35
	0	50	15
	$\sum$	50	50

- What happens in this situation to the prediction?
- At some point, we need to multiply with $p(\text{love} = 1|\text{true}) = 0$
- This leads to a total probability of zero (for this class), irrespective of the other features
  - Even if another feature would be a perfect predictor!

$\rightarrow$ Smoothing (as before)!

Smoothing

- Whenever multiplication is involved, zeros are dangerous
- Smoothing is used to avoid zeros
- Different possibilities
- Simple: Add something to the probabilities
  - $\frac{x_i+1}{N+1}$
  - This leads to values slightly above zero

Summary

- Probability theory
  - Probability: Fraction of positive over all possible events
  - Conditional probability: Restrict the space of possible events
- Naive Bayes
  - Probability-based classification algorithm
  - Assumes feature independence (therefore: “naive”)
  - Still used in many applications
    - E.g., spam classification

References I

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