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LOGISTIC REGRESSION

Sprachverarbeitung (Vorlesung)

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*Based on slides by Nils Reiter

Recap

- So far
  - Two ML algorithms: Naive Bayes, decision tree
  - Feature-based ML: Features interpretable and based on "domain knowledge"
- Naive Bayes
  - Training
    - Calculate $p(\text{FEATURE}|\text{CLASS})$ for all features, feature values and classes
  - Prediction
    - Calculate $p(\text{CLASS}|\text{FEATURES})$, assign class with highest probability
  - Assume feature independence

01

REGRESSION

$$\left.\begin{array}{l}\text{Linear}\\ \text{Logistic}\end{array}\right\}\ \text{Regression}$$

Regression and Neural Networks

- Neural Networks
  - Conceptually developed in the 20th century
  - Mainstream ML method in NLP since 2010
  - Building block of large language models (like ChatGPT)
  - But also a flexible ML algorithm by itself
  - Building block of neural networks: **Logistic regression**

Regression

Linear regression

- Prediction of numeric values (e.g., height and weight)

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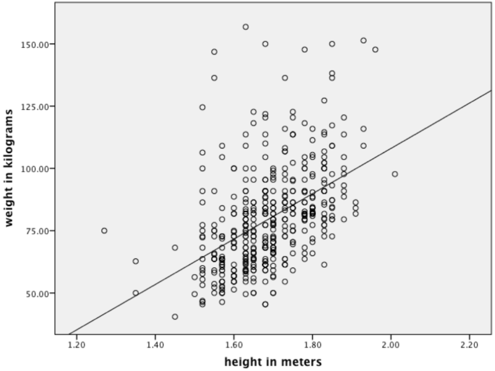

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Regression

Linear regression

- Prediction of numeric values (e.g., height and weight)
- "Linear" regression: Prediction of a linear relation between numeric values (i.e., a line)
- But: many relations are not linear

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Logistic Regression

- Classification algorithm: Instances are grouped into *previously known* classes
- Binary classification: Two classes (e.g., positive/negative)
- Extension of linear regression

Linear/logistic regression in parallel

Linear Regression

Task Setup

- Input (x): A (collection of) numeric feature values
- Output (y): A numeric value

Example

Given the length of a narrative text in number of sentences, predict the number of characters present in its plot

Linear Regression

The data set

x	y (# characters)
10	3
105	5
150	8
210	12
250	7
295	13

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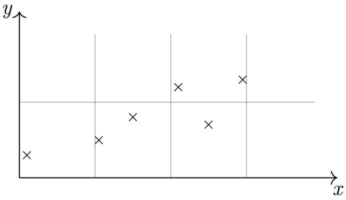

Figure: Data set, each $\times$ represents a text (x: text length, y: num. of characters)

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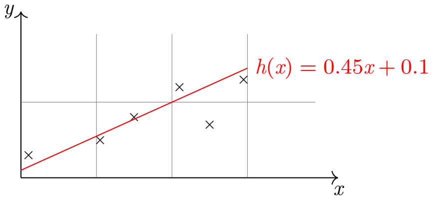
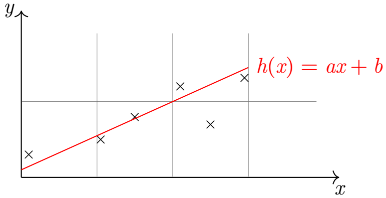


Figure: Data set, each × represents a text (x: text length, y: num. of characters)

Linear Regression



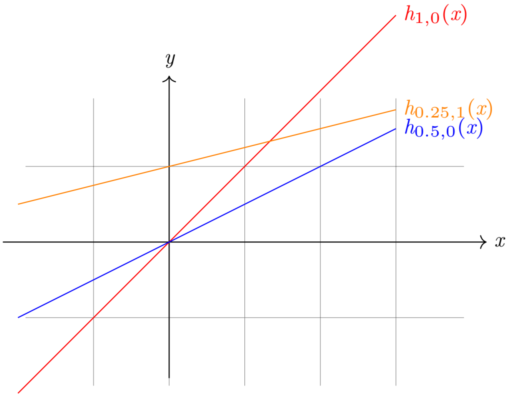
Prediction Model

- Linear regression with one variable (= univariate linear regression)
- Data: (x, y)
- Prediction (hypothesis function): $y = h_{a,b}(x) = ax + b$
- How to set parameters a and b? $\rightarrow$ training algorithm

Linear Regression

Prediction Model

- $h_{a,b}(x) = ax + b$ describes a set of functions
  - $h_{1,0}(x)$ is one concrete function with $h_{1,0}(x) = 1x + 0 = x$



Linear vs. Logistic Regression

- Linear regression: Prediction of numerical data
- Logistic regression: Prediction of (binary) categorical data

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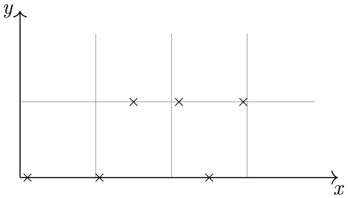
Examples

- Our interest
  - Literature quality
- Given the number of characters in a narrative text
- Will a book win the Nobel prize?
  - Two classes: Yes/No

Logistic Regression

The data set



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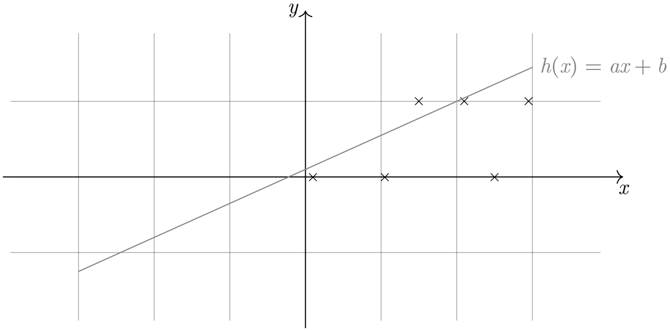
How to predict these values?



linear equation

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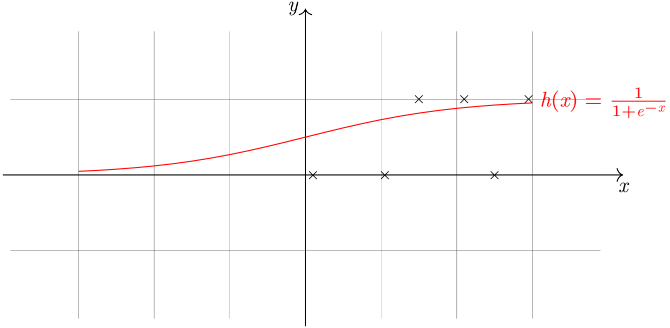
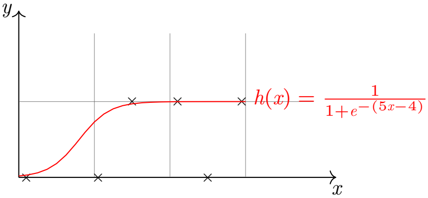


logistic function

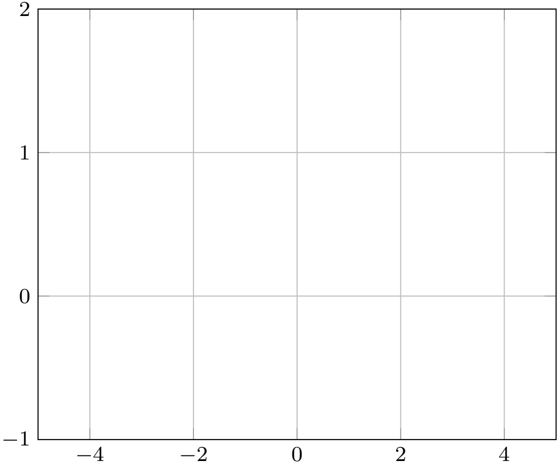
Parameter Fitting



- Linear equations can be wrapped in a logistic one
- Same parameters to be tuned (a and b)
- $e = \sum_{n=0}^{\infty} \frac{1}{n!} = 2.71828$ (Euler's number)

The Logistic Function

$y = \frac{1}{1+e^{-(ax+b)}}$ (general form)



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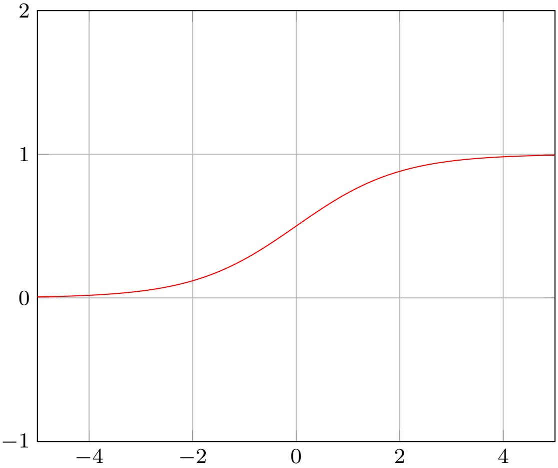

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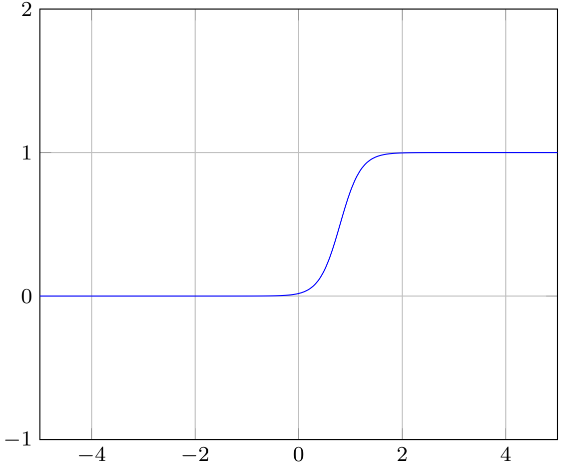

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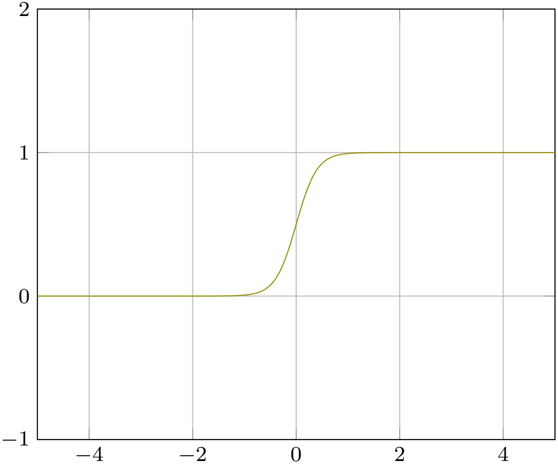

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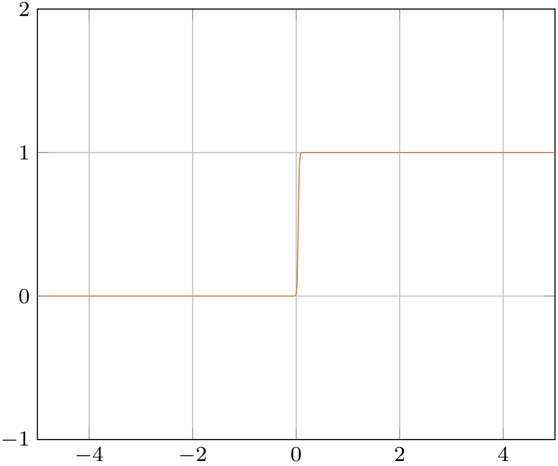

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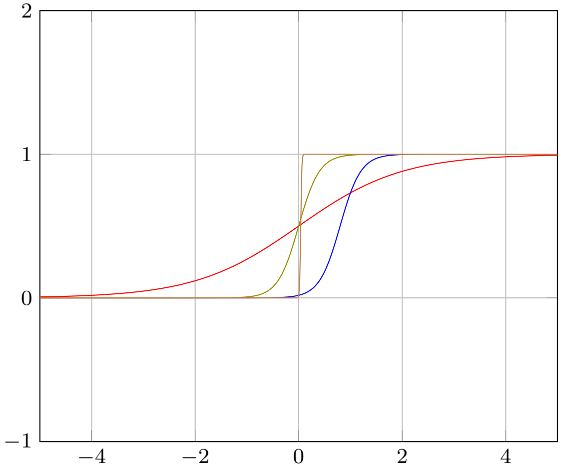

Summary: Logistic Regression (with a single variable)

Logistic regression is half of the math of deep learning

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Logistic regression is half of the math of deep learning

- Logistic Regression: Predicting binary values
- Model
  - Logistic equations
  - $y = \frac{1}{1+e^{-(ax+b)}}$
- Learning algorithm: How to choose a and b?

Learning Regression Models

- How to select the parameters a, b such that the hypothesis function describes the data points as best as possible?
- Learning algorithm Gradient Descent

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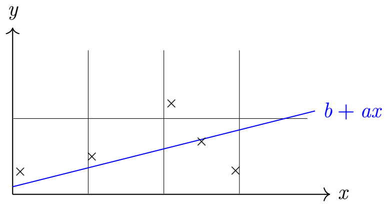
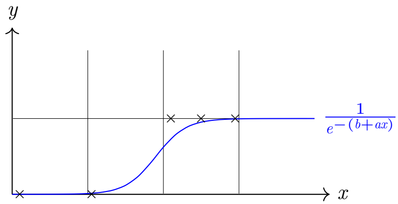
Gradient descent is half of the algorithms of deep learning

Loss: Intuition

The *loss* measures the 'wrongness' of values for a and b.





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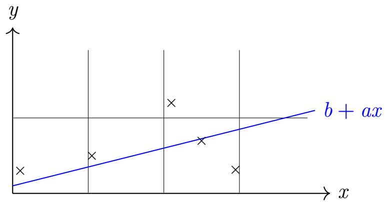

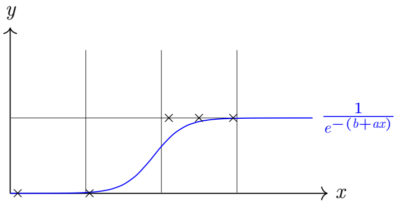

- How big is the gap between a hypothesis and the data?
- Is $(a, b) = (0.3, 0.5)$ or $(a, b) = (0.4, 0.4)$ better?

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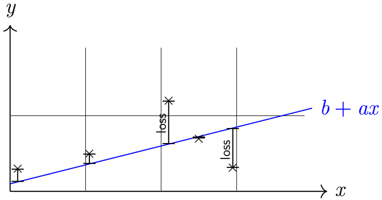

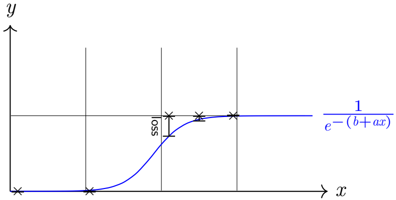

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- Loss should be as small as possible
- Total loss can be calculated for given parameters $\vec{w} = (a, b)$ (and a full data set)
  $\Rightarrow$ I.e.: Loss can be expressed as a function of $\vec{w}$!

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- Loss should be as small as possible
- Total loss can be calculated for given parameters $\vec{w} = (a, b)$ (and a full data set)

  $\Rightarrow$ I.e.: Loss can be expressed as a function of $\vec{w}$!
- Idea:
  - We change $\vec{w}$ until we find the minimum of the function
  - We use the derivative to find out if we are in a minimum
  - The derivative also tells us how to change the update parameters a and b

Loss Function: Intuition

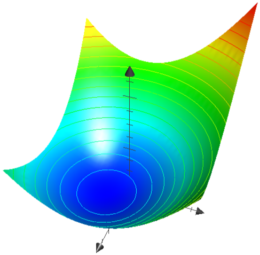
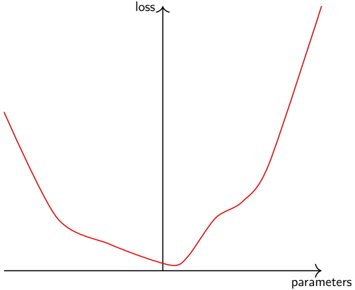


Figure: The loss function with two parameters

Loss function: Intuition



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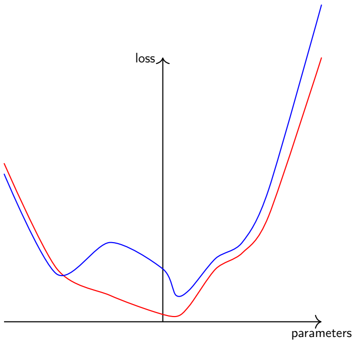

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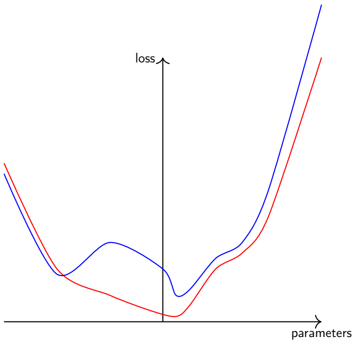

Function should be convex!
If not, we might get stuck in local minimum

Hypothesis vs. Loss Function

- Hypothesis function h
  - Calculates outcomes, given feature values x
- Loss function J
  - Calculates 'wrongness' of h, given parameter values $\vec{w}$ (and a data set)
  - In reality, $\vec{w}$ represents many more parameters (thousands)

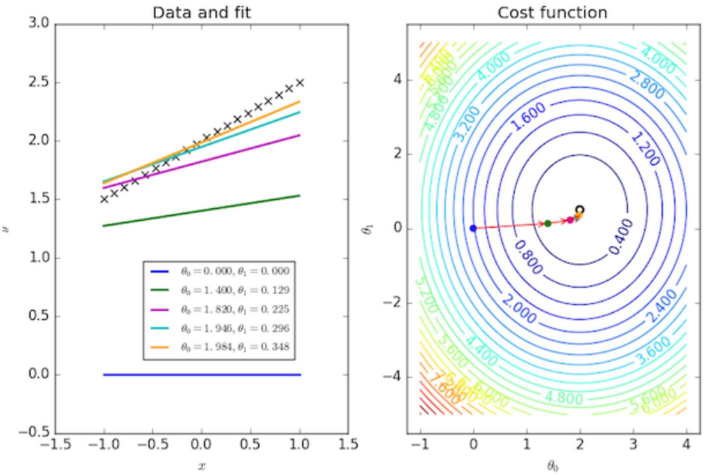

Figure: Visualizing gradient descent Source

Loss Function

Definition

Loss function depends on hypothesis function

Linear hypothesis function

- $h(x) = ax + b$
- Loss: Mean squared error

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Loss Function

Definition

Loss function depends on hypothesis function

Linear hypothesis function

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Logistic hypothesis function

- $h(x) = \frac{1}{e^{-(b+ax)}}$
- Loss: (Binary) cross-entropy loss

Loss Function

Definition for Linear Regression

- The loss function is a function on parameter values a and b
  (for a given hypothesis function and data set)
  - Hypothesis function: $h_{\vec{w}} = w_1 x + w_0$

$\vec{w} = (a, b)$: parameters $\quad h_{\vec{w}}$: hypothesis function $\quad m$: number of items

$$J(\vec{w}) =$$

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$$J(\vec{w}) = \qquad h_{\vec{w}}(x_i) - y_i$$

- Calculate the loss for item i

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$$\vec{w} = (a, b)\text{: parameters} \quad h_{\vec{w}}\text{: hypothesis function} \quad m\text{: number of items}$$

$$J(\vec{w}) = \qquad (h_{\vec{w}}(x_i) - y_i)^2$$

- Calculate the loss for item i
- Square the error

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- Sum them up

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  - Known as: Mean squared error

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$\vec{w} = (a, b)$: parameters $\quad h_{\vec{w}}$: hypothesis function $\quad m$: number of items

$$J(\vec{w}) = \frac{1}{2}\frac{1}{m}\sum_{i=1}^{m}(h_{\vec{w}}(x_i) - y_i)^2$$

- Calculate the loss for item i
- Square the error
- Sum them up
- Divide by the number of items
  - Known as: Mean squared error
- Divide by two
  - out of convenience, because derivation

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Definition for Logistic Regression

- Two cases: $y_i = 0$ or $y_i = 1$ – y_i: real outcome for instance i

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$$J(\vec{w}) = \qquad h_{\vec{w}}(x_i) + \qquad (1 - h_{\vec{w}}(x_i))$$

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Loss function

Definition for Logistic Regression

- Two cases: $y_i = 0$ or $y_i = 1 - y_i$: real outcome for instance i

$$J(\vec{w}) = \qquad \log h_{\vec{w}}(x_i) + \qquad \log\left(1 - h_{\vec{w}}(x_i)\right)$$

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y_i	$h_{\vec{w}}(x_i)$	$y_i \log h_{\vec{w}}(x_i) + (1 - y_i)\log(1 - h_{\vec{w}}(x_i))$
0	1	−23.2535
0	0	0
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$$J(\vec{w}) = -\frac{1}{m}\sum_{i=0}^{m} \underbrace{y_i \log h_{\vec{w}}(x_i)}_{0 \text{ iff } y_i=0} + \underbrace{(1 - y_i)\log(1 - h_{\vec{w}}(x_i))}_{0 \text{ iff } y_i=1}$$

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Caveat: $\log 0$ is undefined
We may need to add something very small

Side note: Log Probabilities

- Relative order is stable: If $a > b$, then $\log a > \log b$
  - No information loss

Side note: Log Probabilities

- Relative order is stable: If $a > b$, then $\log a > \log b$
  - No information loss
- Multiplication turns to addition: $\log(a \cdot b) = \log a + \log b$
  - Addition is much faster than multiplication in a computer
  - Pays off because we're doing this a lot

More Dimensions

- Above: 1 dimension, 2 parameters
  - a: slope, b: y-intercept
  - Input feature x, a single value

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  - $\vec{w} = \langle w_0, w_1, \ldots, w_n \rangle$ (n dimensions)
  - Input vector $\vec{x}$ with $n - 1$ dimensions
  - Hypothesis function: $h_{\vec{w}}(x) = w_n x_n + w_{n-1} x_{n-1} + \ldots w_1 x_1 + w_0$
    - w_0: y-intercept, w_1 to w_n: slopes

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  - a: slope, b: y-intercept
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  - Input vector $\vec{x}$ with $n - 1$ dimensions
  - Hypothesis function: $h_{\vec{w}}(x) = w_n x_n + w_{n-1} x_{n-1} + \ldots w_1 x_1 + w_0$
    - w_0: y-intercept, w_1 to w_n: slopes
- Algorithms
  - Derivatives more complicated
  - Otherwise identical

02

SUMMARY

Summary

Regression

- Fitting parameters to a data distribution
  - Linear Regression : Numeric prediction algorithm
    - Prediction model: $h_{\vec{w}}(x) = ax + b$
  - Logistic Regression: Classification algorithm
    - Prediction model: $h_{\vec{w}}(x) = \frac{1}{e^{-(b+ax)}}$
- Training algorithm: Gradient descent

Gradient Descent

- Initialise $\vec{w}$ with random values (e.g., 0)
- Repeat:
  - Find the direction to the minimum by taking the derivative
  - Change $\vec{w}$ accordingly, using a learning rate η
  - Stop when $\vec{w}$ don't change anymore